

# A limit theorem for singular stochastic differential equations

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**Abstract** We study the weak limits of solutions to SDEs

$$dX_n(t) = a_n(X_n(t)) dt + dW(t),$$

where the sequence  $\{a_n\}$  converges in some sense to  $(c_- \mathbb{1}_{x<0} + c_+ \mathbb{1}_{x>0})/x + \gamma \delta_0$ . Here  $\delta_0$  is the Dirac delta function concentrated at zero. A limit of  $\{X_n\}$  may be a Bessel process, a skew Bessel process, or a mixture of Bessel processes.

**Keywords** Bessel process, skew Bessel process, limit theorems

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## 1 Introduction

Consider the stochastic differential equation

$$dX(t) = a(X(t)) dt + dW(t), \quad t \geq 0, \quad (1)$$

where  $a$  is a locally integrable function.

The aim of this paper is to study convergence in distribution of the sequence of processes  $\{X(nt)/\sqrt{n}, t \geq 0\}$  as  $n \rightarrow \infty$ .

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Observe that

$$dX_n(t) = \sqrt{na}(\sqrt{n}X_n(t)) dt + dW_n(t), \quad t \geq 0,$$

where  $W_n(t) = W(nt)/\sqrt{n}$ ,  $t \geq 0$  is a Wiener process, and  $X_n(t) = X(nt)/\sqrt{n}$ ,  $t \geq 0$ .

Hence, to study the sequence  $\{X(nt)/\sqrt{n}\}$ , it suffices to investigate the SDEs

$$dX_n(t) = a_n(X_n(t)) dt + dW(t), \quad t \geq 0,$$

where  $a_n(x) = na(nx)$ .

If  $a \in L_1(\mathbb{R})$ , then  $a_n$  converges in generalized sense to  $\alpha\delta_0$ , where  $\delta_0$  is the Dirac delta function at zero, where  $\alpha = \int_{\mathbb{R}} a(x) dx$ . It is well known that in this case the sequence  $\{X_n\}$  converges weakly to a skew Brownian motion with parameter  $\gamma = \text{th}(\alpha) = \frac{e^\alpha - e^{-\alpha}}{e^\alpha + e^{-\alpha}}$ ; see, for example, [14, 10]. Recall that [5, 10] the skew Brownian motion  $W_\gamma(t)$  with parameter  $\gamma$ ,  $|\gamma| \leq 1$ , is a unique (strong) solution to the SDE

$$dW_\gamma(t) = dW(t) + \gamma dL_{W_\gamma}^0(t),$$

where  $L_{W_\gamma}^0(t) = \lim_{\varepsilon \rightarrow 0+} (2\varepsilon)^{-1} \int_0^t \mathbb{1}_{|W_\gamma(s)| \leq \varepsilon} ds$  is the local time of the process  $W_\gamma$  at 0. The process  $W_\gamma$  is a continuous Markov process with transition probability density function  $p_t(x, y) = \varphi_t(x-y) + \gamma \text{sign}(y) \varphi_t(|x|+|y|)$ ,  $x, y \in \mathbb{R}$ , where  $\varphi_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$  is the density of the normal distribution  $N(0, t)$ . Note also that  $W_\gamma$  can be obtained from excursions of a Wiener process pointing them (independently of each other) up and down with probabilities  $p = (1+\gamma)/2$  and  $q = (1-\gamma)/2$ , respectively.

Kulinich et al. [8, 7] considered limit theorems in the case where  $a$  is nonintegrable function such that

$$\lim_{x \rightarrow \pm\infty} \frac{1}{x} \int_0^x |va(v) - c_\pm| dv = 0, \quad |xa(x)| \leq C, \quad (2)$$

where  $c_\pm > -1/2$  are constants. In this case,  $a_n(x)$  converges in some sense to  $c_- \mathbb{1}_{x < 0} + c_+ \mathbb{1}_{x \geq 0}$  as  $n \rightarrow \infty$ .

For instance, if  $a(x) = c_\pm/x$  for  $\pm x > x_0$ , then, for  $c_- < 1/2 < c_+$ , the sequence  $X_n$  converges weakly to a Bessel process. If  $c_- = c_+ > -1/2$ , then  $|X_n|$  also converges weakly to a Bessel process. The problem of weak convergence of  $X_n$  for (e.g.)  $c_- = c_+ > -1/2$  or  $c_- < c_+ \leq 1/2$  was not considered.

In this paper, we generalize the results of [14, 8] to the case

$$a(x) = \tilde{a}(x) + \frac{\bar{c}(x)}{x}, \quad x \in \mathbb{R},$$

where  $\tilde{a}$  is integrable on  $(-\infty; \infty)$ , and

$$\bar{c}(x) = c_+ \cdot \mathbb{1}_{x > 1} + c_- \cdot \mathbb{1}_{x < -1}, \quad x \in \mathbb{R}.$$

We consider all possible limit processes (depending on  $c_+$  and  $c_-$ ). In particular, we show that, for  $c_+ = c_- < 1/2$ , the limit process is a skew Bessel process (see Section 2).

## 2 Bessel process. Skew Bessel process. Definition, properties

We recall the definition and some properties of Bessel processes.

Let  $\delta \geq 0$  and  $x_0 \in \mathbb{R}$ . Consider the SDE

$$Z(x_0^2, t) = x_0^2 + 2 \int_0^t \sqrt{|Z(x_0^2, s)|} dW(s) + \delta t, \quad t \geq 0, \quad (3)$$

where  $W$  is a Wiener process.

It is known (see [15], XI.1, (1.1)), that there exists a unique strong solution  $Z(x_0^2, \cdot)$  of (3). This solution is called the squared  $\delta$ -dimensional Bessel process. The process  $Z(x_0^2, \cdot)$  is nonnegative a.s.

**Definition 1.** The process  $B_c(x_0, t) = \sqrt{Z(x_0^2, t)}$  with  $x_0 \geq 0$  is called the (non-negative) *Bessel process* with parameter  $c = (\delta - 1)/2$ .

We will call the process  $B_c^-(x_0, t) = -B_c(x_0, t) = -\sqrt{Z(x_0^2, t)}$  with  $x_0 \leq 0$  the nonpositive Bessel process.

Recall the following properties of the Bessel process (see [15, Chap. XI]).

The Bessel process  $\xi(t) = B_c(x_0, t)$  satisfies the SDE

$$d\xi_t = dW_t + \frac{c}{\xi_t} dt, \quad t < T_0,$$

where  $T_0$  is the first hitting time of 0. If  $\delta \geq 2$  (i.e.,  $c \geq 1/2$ ), then the Bessel process with probability 1 does not hit 0.

If  $0 < \delta < 2$  (i.e.,  $-1/2 < c < 1/2$ ), then with probability 1 the Bessel process hits 0 but spends zero time at 0. In particular, if  $\delta = 1$  (i.e.,  $c = 0$ ), then the Bessel process is a reflecting Brownian motion.

If  $\delta = 0$  (i.e.,  $c = -1/2$ ), then with probability 1 the process attains 0 and remains there forever.

The scale function of the Bessel process  $B_c$  equals

$$\psi_c(x) = \begin{cases} -x^{-2c+1} & \text{if } c > 1/2, \\ \ln x & \text{if } c = 1/2, \\ x^{-2c+1} & \text{if } c < 1/2, \end{cases} \quad (4)$$

that is,

$$P_x(T_a < T_b) = \frac{\psi_c(b) - \psi_c(x)}{\psi_c(b) - \psi_c(a)} \quad \text{for any } 0 < a < x < b,$$

where  $T_y = \inf\{t \geq 0 : B_c(t) = y\}$ .

The transition density for  $c > -1/2$ ,  $x, y > 0$ , and  $t > 0$  equals

$$p_t^c(x, y) = t^{-1} (y/x)^\nu y \exp(-(x^2 + y^2)/2t) I_\nu(xy/t),$$

where  $I_\nu$  is a Bessel function of index  $\nu = c - 1/2$ .

Let  $c \in (-1/2, 1/2)$ , and let  $p_t^{0,c}(x, y)$  be the transition density of the Bessel process  $B_c$  killed at 0.

Set

$$p_t^{skew}(x, y) = p_t^{0,c}(|x|, |y|) \cdot \mathbb{1}_{xy > 0} + \frac{1 + \gamma \operatorname{sign} y}{2} (p_t^c(|x|, |y|) - p_t^{0,c}(|x|, |y|)), \quad x, y \in \mathbb{R}.$$

It is easy to verify that this function satisfies the Chapman–Kolmogorov equation, is nonnegative, and  $\int_{\mathbb{R}} p_t^{skew}(x, y) dy = 1$ ,  $x \in \mathbb{R}$ .

**Definition 2.** A time-homogeneous Markov process with the transition density  $p_t^{skew}$  is called the *skew Bessel process*  $B_{c,\gamma}^{skew}$  with parameters  $c$  and  $\gamma \in [-1, 1]$ .

*Remark 1.* We do not consider the skew Bessel process for  $c \geq 1/2$  because  $B_c(x_0, \cdot)$  does not hit 0 if  $x_0 \neq 0$ .

*Remark 2.* The skew Bessel process  $B^{skew}$  can be obtained from a nonnegative Bessel process by pointing its excursions up with probability  $p = \frac{1+\gamma}{2}$  and down with probability  $q = \frac{1-\gamma}{2}$ , similarly to the case of a skew Brownian motion; see arguments in [1], Section 2.

Thus, the scale function of the skew Bessel process equals

$$\psi_{skew}(x) = (q\mathbb{1}_{x \geq 0} - p\mathbb{1}_{x < 0})|x|^{-2c+1}, \quad x \in \mathbb{R}. \quad (5)$$

For other properties of the skew Bessel process, we refer to [2].

*Remark 3.* If  $x_0 > 0$  and  $p = 1$  (i.e.,  $\gamma = 1$ ), then  $B_{c,\gamma}^{skew}(x_0, \cdot)$  is a (nonnegative) Bessel process  $B_c(x_0, \cdot)$  with parameter  $c$ :  $B_{c,1}^{skew}(x_0, \cdot) \stackrel{d}{=} B_c(x_0, \cdot)$ .

Also, the absolute value of the skew Bessel process  $|B_{c,\gamma}^{skew}|$  is a (nonnegative) Bessel process  $B_c(x_0, \cdot)$  with parameter  $c$ :  $|B_{c,\gamma}^{skew}(x_0, \cdot)| \stackrel{d}{=} B_c(x_0, \cdot)$ .

If  $c = 0$ , then  $B_{c,\gamma}^{skew}$  is a skew Brownian motion:  $B_{0,\gamma}^{skew}(\cdot) \stackrel{d}{=} W_\gamma(\cdot)$ .

### 3 Main results

Let

$$a(x) = \tilde{a}(x) + \frac{\bar{c}(x)}{x}, \quad x \in \mathbb{R},$$

where  $\tilde{a} \in L_1(\mathbb{R})$  and

$$\bar{c}(x) = c_+ \cdot \mathbb{1}_{x > 1} + c_- \cdot \mathbb{1}_{x < -1}, \quad x \in \mathbb{R}.$$

Let  $X_n(t)$ ,  $t \geq 0$ , be the solution of the SDE

$$\begin{cases} dX_n(t) = na(nX_n(t)) dt + dW(t) \\ \quad = \left( n\tilde{a}(nX_n(t)) + \frac{\bar{c}(nX_n(t))}{X_n(t)} \right) dt + dW(t), \quad t \geq 0, \\ X_n(0) = x_0. \end{cases}$$

The existence and uniqueness of a strong solution of this SDE follows from [3, Thm. 4.53].

**Theorem 1.** *If  $c_+$  and  $c_- > -1/2$ , then the sequence of processes  $\{X_n\}$  converges weakly to a limit process  $X_\infty$ . In particular:*

A1. *If*

- (a)  $x_0 > 0$  and  $c_+ \geq 1/2$ , or
- (b)  $x_0 \geq 0$  and  $c_- < c_+ < 1/2$ , or
- (c)  $x_0 = 0$  and  $c_- < 1/2 \leq c_+$ ,

*then*

$$X_\infty(t) = B_{c_+}^+(x_0, t), \quad t \geq 0.$$

A2. *Similarly, if*

- (a)  $x_0 < 0$  and  $c_- \geq 1/2$ , or
- (b)  $x_0 \leq 0$  and  $c_+ < c_- < 1/2$ , or
- (c)  $x_0 = 0$  and  $c_+ < 1/2 \leq c_-$ ,

*then*

$$X_\infty(t) = B_{c_-}^-(x_0, t), \quad t \geq 0.$$

A3. *If  $x_0 < 0$ ,  $c_- < 1/2$ , and  $c_- < c_+$ , then the limiting process evolves as  $B_{c_-}^-$  until hitting 0 and then proceeds as  $B_{c_+}^+$  indefinitely, that is,*

$$X_\infty(t) = B_{c_-}^-(x_0, t) \cdot \mathbb{1}_{t \leq \tau} + B_{c_+}^+(0, t - \tau) \cdot \mathbb{1}_{t > \tau}, \quad t \geq 0,$$

*where  $\tau = \inf\{t: X_\infty(t) \geq 0\}$  and  $B_{c_\pm}^\pm$  are independent (positive and negative) Bessel processes.*

A4. *Similarly, if  $x_0 > 0$ ,  $c_+ < 1/2$ , and  $c_+ < c_-$ , then*

$$X_\infty(t) = B_{c_+}^+(x_0, t) \cdot \mathbb{1}_{t \leq \tau} + B_{c_-}^-(0, t - \tau) \cdot \mathbb{1}_{t > \tau}, \quad t \geq 0,$$

*where  $\tau = \inf\{t: X_\infty(t) \leq 0\}$ .*

A5. *If  $c_+ = c_- =: c < 1/2$ , then, for any  $x_0$ ,*

$$X_\infty(t) = B_{c, \gamma}^{skew}(x_0, t), \quad t \geq 0,$$

$$\text{where } \gamma = \text{th}(\int_{-\infty}^{+\infty} \tilde{a}(z) dz) = \frac{1 - \exp\{-2 \int_{-\infty}^{+\infty} \tilde{a}(z) dz\}}{1 + \exp\{-2 \int_{-\infty}^{+\infty} \tilde{a}(z) dz\}}.$$

A6. *Finally, if  $x_0 = 0$ ,  $c_+ \geq 1/2$ , and  $c_- \geq 1/2$ , then the distribution of the limit process  $X_\infty$  equals*

$$p \cdot \mathbb{P}_{B_{c_+}^+} + (1 - p) \cdot \mathbb{P}_{B_{c_-}^-},$$

*where*

$$p = \frac{\int_0^\infty A(-y)(y \vee 1)^{-2c_-} dy}{\int_0^\infty (A(-y)(y \vee 1)^{-2c_-} + A(y)(y \vee 1)^{-2c_+}) dy}, \quad (6)$$

*$A(y) = \exp\{-2 \int_0^y \tilde{a}(z) dz\}$ , and  $\mathbb{P}_{B_{c_\pm}^\pm}$  are the distributions of positive and negative Bessel processes  $B_{c_\pm}^\pm(0, \cdot)$  starting from 0.*

*Remark 4.* Some results of the theorem follow from [8]. However, we apply here the general approach applicable to all cases simultaneously. Condition (2) is somewhat weaker than  $\tilde{a} \in L_1(\mathbb{R})$ . However, we do not assume that  $\sup_x |x\tilde{a}(x)| < \infty$ , contrary to the paper [8].

#### 4 Proof

It follows from [9, Section 3] or [11, Section 3.7] that if A1 is satisfied, then, for any  $\alpha > 0$ , we have the convergence

$$X_n(\cdot \wedge \tau^{n,\alpha}) \Rightarrow B_{c+}^+(x_0, \cdot \wedge \tau^{0,\alpha}), \quad n \rightarrow \infty,$$

where  $\tau^{n,\alpha} = \inf\{t \geq 0: X_n(t) \leq \alpha\}$  and  $\tau^{0,\alpha} = \inf\{t \geq 0: B_{c+}^+(x_0, t) \leq \alpha\}$ . Since the process  $B_{c+}^+(x_0, \cdot)$  does not hit 0, this yields the proof. Case A2 is considered similarly.

To prove all other items of Theorem 1, we use the method proposed in [13].

Let  $\{\xi^{(n)}, n \geq 0\}$  be a sequence of continuous homogeneous strong Markov processes. For  $\alpha > 0$ , set

$$\tau^{n,\alpha} := \inf\{t \geq 0: |\xi^{(n)}(t)| \leq \alpha\}, \quad \sigma^{n,\alpha} := \inf\{t \geq 0: |\xi^{(n)}(t)| \geq \alpha\}.$$

We denote by  $\xi_{x_0}^{(n)}$  a process that has the distribution of  $\xi^{(n)}$  conditioned by  $\xi^{(n)}(0) = x_0$ .

The next statement is a particular case of Theorem 2 of [13].

**Lemma 1.** Assume that the sequence  $\{\xi^{(n)}, n \geq 0\}$  satisfies the following conditions:

$$\xi^{(n)}(0) \Rightarrow \xi^{(0)}(0); \quad (7)$$

$$\forall T > 0 \forall \varepsilon > 0 \exists \delta > 0 \exists n_0 \forall n \geq n_0$$

$$P\left(\sup_{\substack{|s-t|<\delta, \\ s,t \in [0,T]}} |\xi^{(n)}(t) - \xi^{(n)}(s)| \geq \varepsilon\right) \leq \varepsilon; \quad (8)$$

$$\forall T > 0 \quad \lim_{\varepsilon \rightarrow 0+} \sup_n E \int_0^T \mathbb{1}_{|\xi^{(n)}(t)| \leq \varepsilon} dt = 0; \quad (9)$$

$$\int_0^\infty \mathbb{1}_{\xi^{(0)}(t)=0} dt = 0 \quad a.s. \quad (10)$$

Assume that, for any  $\alpha > 0$ ,  $x_0 \in \mathbb{R}$ , and any sequence  $\{x_n\}$  such that  $\lim_{n \rightarrow \infty} x_n = x_0$ , we have

$$(\xi_{x_n}^{(n)}(\cdot \wedge \tau^{n,\alpha}), \tau^{n,\alpha}) \Rightarrow (\xi_{x_0}^{(0)}(\cdot \wedge \tau^{0,\alpha}), \tau^{0,\alpha}), \quad n \rightarrow \infty; \quad (11)$$

$$\xi_{x_n}^{(n)}(\sigma^{n,\alpha}) \Rightarrow \xi_{x_0}^{(0)}(\sigma^{0,\alpha}), \quad n \rightarrow \infty. \quad (12)$$

Then  $\xi^{(n)} \Rightarrow \xi^{(0)}$  in  $C([0, \infty))$  as  $n \rightarrow \infty$ .

We apply this lemma for  $\xi^{(n)} = X_n, n \geq 1$ , and  $\xi^{(0)} = X_\infty$  in cases A1–A5 of the theorem. Case A6 will be considered separately.

**Remark 5.** Condition (12) is the only condition that is not true in case A6. It fails if  $x_0 = 0$ . Indeed, for any  $x > 0$ , the process  $B_{+}^+(x, \cdot)$  does not hit 0. So, we may select a sequence  $\{x_n\} \subset (0, \infty)$  that converges to 0 sufficiently slowly and such that, given  $X_n(0) = x_n$ , we have  $X_n(\cdot) \Rightarrow B_+(0, \cdot)$  and  $\lim_{n \rightarrow \infty} P(\exists t \geq 0: X_n(t) = 0) = 0$ . The concrete selection of  $\{x_k\}$  can be done using formulas (15) and (16). Since  $B_+(0, \sigma^{0,\alpha}) = \alpha$  a.s., we get  $X_n(\sigma^{n,\alpha}) \Rightarrow \alpha$ . However, if all  $x_n$  were negative, then the limit might be  $-\alpha$ .

Conditions (7) and (10) are obvious.

The convergence

$$\forall \alpha > 0 \quad \xi_{x_n}^{(n)}(\cdot \wedge \tau^{n,\alpha}) \Rightarrow \xi_{x_0}^{(0)}(\cdot \wedge \tau^{0,\alpha}), \quad n \rightarrow \infty, \quad (13)$$

follows from [9, Section 3] or [11, Section 3.7]. Since

$$P(\forall \varepsilon > 0 \exists t \in (\tau^{0,\alpha}, \tau^{0,\alpha} + \varepsilon): |\xi_{x_0}^{(0)}(t)| < \alpha \mid \tau^{0,\alpha} < \infty) = 1,$$

convergence (13) yields the convergence of pairs (11).

Let us check condition (8). Set

$$\begin{aligned} A(y) &= \exp \left\{ -2 \int_0^y \tilde{a}(z) dz \right\}, \\ A_n(y) &= \exp \left\{ -2 \int_0^y n \tilde{a}(nz) dz \right\} = A(ny), \quad y \in \mathbb{R}, \\ \Phi_n(x) &= \int_0^x A_n(y) dy, \quad x \in \mathbb{R}. \end{aligned}$$

Observe that  $\Phi_n : \mathbb{R} \rightarrow \mathbb{R}$  is a bijection,  $\Phi_n(0) = 0$ , and

$$\exists L > 0 \quad \forall n \quad \forall x, y \in \mathbb{R} \quad L^{-1}|x - y| \leq |\Phi_n(x) - \Phi_n(y)| \leq L|x - y|.$$

Itô's formula yields

$$d\Phi_n(X_n(t)) = A(nX_n(t)) \left( \frac{\bar{c}(nX_n(t))}{X_n(t)} dt + dW(t) \right).$$

So

$$\begin{aligned} |X_n(t) - X_n(s)| &\leq L |\Phi_n(X_n(t)) - \Phi_n(X_n(s))| \\ &\leq L \left| \int_s^t A(nX_n(z)) \frac{\bar{c}(nX_n(z))}{X_n(z)} dz \right| + L \left| \int_s^t A(nX_n(z)) dW(z) \right|. \end{aligned}$$

Let  $|s - t| < \delta$ , and let  $\Delta > 0$  be fixed. Denote  $f_n(t) := \int_0^t A(nX_n(z)) dW(z)$ .

a) Assume that  $|X_n(z)| > \Delta, z \in [s, t]$ . Then

$$\left| \int_s^t A(nX_n(z)) \frac{\bar{c}(nX_n(z))}{X_n(z)} dz \right| \leq C\delta/\Delta,$$

where  $C = \|A\|_\infty \max(|c_-|, |c_+|) < \infty$ . Hence, we have the estimate

$$|X_n(t) - X_n(s)| \leq LC\delta/\Delta + L\omega_{f_n}(\delta),$$

where  $\omega_f(\delta) = \sup_{|s-t|<\delta, s,t \in [0,T]} |f(t) - f(s)|$  is the modulus of continuity.

b) Assume that  $|X_n(z_0)| \leq \Delta$  for some  $z_0 \in [s, t]$ .

Denote  $\tau := \inf\{z \geq s: |X_n(z)| \leq \Delta\}$  and  $\sigma := \sup\{z \leq t: |X_n(z)| \leq \Delta\}$ . Then

$$\begin{aligned} |X_n(t) - X_n(s)| &\leq |X_n(s) - X_n(\tau)| + |X_n(\sigma) - X_n(t)| + 2\Delta \\ &\leq 2LC\delta/\Delta + 2L\omega_{f_n}(\delta) + 2\Delta. \end{aligned}$$

Thus, in any case, we have the following estimate of the modulus of continuity:

$$\omega_{X_n}(\delta) \leq 2LC\delta/\Delta + 2L\omega_{f_n}(\delta) + 2\Delta.$$

Let  $\Delta \leq \varepsilon/6$ . Then, for  $\delta \leq \frac{\varepsilon\Delta}{6LC}$ , we have

$$\begin{aligned} \sup_n \mathbb{P}(\omega_{X_n}(\delta) \geq \varepsilon) &\leq \sup_n \mathbb{P}(\varepsilon/3 + 2L\omega_{f_n}(\delta) + \varepsilon/3 \geq \varepsilon) \\ &= \sup_n \mathbb{P}(\omega_{f_n}(\delta) \geq \varepsilon/6L) \rightarrow 0, \quad \delta \rightarrow 0+. \end{aligned}$$

The last convergence follows from the fact that the sequence of distributions of  $\{f_n(\cdot) = \int_0^\cdot A(nX_n(z)) dW(z)\}_{n \geq 1}$  in the space of continuous functions is weakly relatively compact because the function  $A$  is bounded.

Let us prove (12) in cases A1–A5.

*Remark 6.* The proof below yields that condition (12) is true if  $x_n = 0$  for all  $n \geq 0$ .

Let  $|x| < \alpha$ . It is easy to see that  $P_x(\sigma^{n,\alpha} < \infty) = 1$ ,  $n \in \mathbb{N} \cup \{\infty\}$ . Since the process  $X_n$  is continuous, we have  $|X_n(\sigma^{n,\alpha})| = \alpha$  a.s.

By  $p_x^n = P_x(X_n(\sigma^{n,\alpha}) = \alpha)$ ,  $n \in \mathbb{N} \cup \{\infty\}$ , we denote the probability to reach  $\alpha$  before reaching  $-\alpha$  when starting from  $x$ .

Using formulas (4) and (5) for the scale of a Bessel process and a skew Bessel process, it is easy to check that

$$p_x^\infty = \begin{cases} \mathbb{1}_{x \geq 0} - \left(1 - \frac{\psi_{c-}(-x)}{\psi_{c-}(\alpha)}\right) \mathbb{1}_{x < 0} & \text{in cases A1, A3,} \\ \frac{\psi_{c+}(x)}{\psi_{c+}(\alpha)} \cdot \mathbb{1}_{x > 0} & \text{in cases A2, A4,} \\ \frac{\psi_c(|x|)}{\psi_c(\alpha)} (q \mathbb{1}_{x \geq 0} - p \mathbb{1}_{x < 0}) + p & \text{in case A5,} \end{cases} \quad (14)$$

where  $\psi_c$  is given in (4).

For  $n \in \mathbb{N}$ , we have (see [4, Section 15] and [15])

$$p_x^n = \frac{\varphi_n(x) - \varphi_n(-\alpha)}{\varphi_n(\alpha) - \varphi_n(-\alpha)}, \quad (15)$$

where

$$\begin{aligned} \varphi_n(x) &= \int_0^x \exp\left\{-2 \int_0^y a_n(z) dz\right\} dy = \int_0^x \exp\left\{-2 \int_0^y na(nz) dz\right\} dy \\ &= \frac{1}{n} \int_0^{nx} \exp\left\{-2 \int_0^y a(z) dz\right\} dy = \frac{1}{n} \varphi(nx), \\ \varphi(x) &:= \int_0^x \exp\left\{-2 \int_0^y a(z) dz\right\} dy. \end{aligned} \quad (16)$$



The function  $\varphi$  is increasing. It follows from the definition of  $a$  that  $\varphi$  is bounded from above (below) iff  $c_+ > 1/2$  ( $c_- > 1/2$ ). The function  $\varphi$  has the following asymptotic behavior:

$$\varphi(x) \sim \begin{cases} \pm A(\pm\infty) \frac{|x|^{1-2c_{\pm}}}{1-2c_{\pm}} & \text{if } c_{\pm} < 1/2, \\ \pm A(\pm\infty) \ln |x| & \text{if } c_{\pm} = 1/2, \end{cases} \quad x \rightarrow \pm\infty, \quad (17)$$

where

$$A(y) = \exp \left\{ -2 \int_0^y \tilde{a}(z) dz \right\}, \quad y \in \mathbb{R},$$

and

$$\lim_{x \rightarrow \pm\infty} \varphi(x) = \varphi(\pm\infty) \in \mathbb{R} \quad \text{if } c_{\pm} > 1/2. \quad (18)$$

Condition (12) follows from (14), (15), (16), (17), (18) in cases A1–A5 (and in case A6 if  $x_n = 0$ ,  $n \geq 0$ ).

Set  $\tau_n = \inf\{t \geq 0: |X_n(t)| \geq 1\}$ .

**Lemma 2.** Assume that

$$\lim_{\varepsilon \rightarrow 0+} \sup_{|x| \leq 1} \sup_n \mathbb{E}_x \int_0^{\tau_n} \mathbb{1}_{|X_n(t)| \leq \varepsilon} dt = 0. \quad (19)$$

Then (9) is satisfied, that is,

$$\forall T > 0 \quad \lim_{\varepsilon \rightarrow 0+} \sup_n \mathbb{E} \int_0^T \mathbb{1}_{|X_n(t)| \leq \varepsilon} dt = 0.$$

**Proof.** Introduce the notations

$$\begin{aligned} S_{n,\pm}^0 &:= 0, & T_{n,\pm}^k &:= \inf\{t \geq S_{n,\pm}^{k-1}: |X_n(t)| = 1\}, \\ S_{n,\pm}^k &:= \inf\{t \geq T_{n,\pm}^k: |X_n(t)| = \varepsilon\}, \\ \tilde{T}_{n,\pm}^k &:= \inf\{t \geq S_{n,\pm}^k: |X_n(t)| = 1\}, \\ \beta_{n,\pm}^k &:= \int_{S_{n,\pm}^k}^{\tilde{T}_{n,\pm}^k} \mathbb{1}_{|X_n(t)| \leq \varepsilon} dt, & \alpha_{n,\pm}^k &:= S_{n,\pm}^k - T_{n,\pm}^k, \quad k \geq 1. \end{aligned}$$

Then

$$\begin{aligned} & \int_0^T \mathbb{1}_{|X_n(t)| \leq \varepsilon} dt \\ & \leq \int_0^{\tau_n} \mathbb{1}_{|X_n(t)| \leq \varepsilon} dt + \sum_k (\beta_{n,+}^1 + \cdots + \beta_{n,+}^k) \mathbb{1}_{\alpha_{n,+}^1 < T, \dots, \alpha_{n,+}^k < T, \alpha_{n,+}^{k+1} \geq T} \\ & \quad + \sum_k (\beta_{n,-}^1 + \cdots + \beta_{n,-}^k) \mathbb{1}_{\alpha_{n,-}^1 < T, \dots, \alpha_{n,-}^k < T, \alpha_{n,-}^{k+1} \geq T}. \end{aligned}$$

It follows from the strong Markov property that

$$\sum_k \mathbb{E} (\beta_{n,+}^1 + \cdots + \beta_{n,+}^k) \mathbb{1}_{\alpha_{n,+}^1 < T, \dots, \alpha_{n,+}^k < T, \alpha_{n,+}^{k+1} \geq T}$$

$$= \sum_k k E_\varepsilon \int_0^{\tau_n} \mathbb{1}_{|X_n(t)| \leq \varepsilon} dt (1 - p_{n,+})^k p_{n,+} = (p_{n,+})^{-1} E_\varepsilon \int_0^{\tau_n} \mathbb{1}_{|X_n(t)| \leq \varepsilon} dt,$$

where  $p_{n,+} = P_1(S_{n,+}^1 \geq T)$ .

Considering the last term similarly, we get the inequality

$$E \int_0^T \mathbb{1}_{|X_n(t)| \leq \varepsilon} dt \leq (1 + (p_{n,+})^{-1} + (p_{n,-})^{-1}) \sup_{|x| \leq 1} \sup_n E_x \int_0^{\tau_n} \mathbb{1}_{|X_n(t)| \leq \varepsilon} dt.$$

It is not difficult to see that  $\sup_n (p_{n,\pm})^{-1} < \infty$ . The lemma is proved.  $\square$

Let us verify (19). It is known [6, Chap. 4.3] that

$$u_{n,\varepsilon}(x) := E_x \int_0^{\tau_n} \mathbb{1}_{|X_n(t)| \leq \varepsilon} dt$$

is of the form

$$u_{n,\varepsilon}(x) = \int_{-1}^1 G_n(x, y) \mathbb{1}_{|y| \leq \varepsilon} m_n(dy), \quad (20)$$

where Green's function  $G_n$  equals

$$G_n(x, y) = \begin{cases} \frac{(\varphi_n(x) - \varphi_n(-1))(\varphi_n(1) - \varphi_n(y))}{\varphi_n(1) - \varphi_n(-1)}, & x \leq y, \\ G_n(y, x), & x \geq y, \end{cases}$$

with  $\varphi_n$  given by formula (16), and

$$m_n(dx) = \exp \left\{ 2 \int_0^x a_n(z) dz \right\} dx.$$

The function  $u_{n,\varepsilon}(x)$  is a generalized solution (because  $a_n$  may be discontinuous) of the equation

$$1/2 u_{n,\varepsilon}''(x) + a_n(x) u_{n,\varepsilon}'(x) = -\mathbb{1}_{|x| \leq \varepsilon}(x), \quad |x| \leq 1,$$

with boundary conditions  $u_{n,\varepsilon}(\pm 1) = 0$ .

A direct verification of the condition  $\lim_{\varepsilon \rightarrow 0+} \sup_{|x| \leq 1} \sup_n u_{n,\varepsilon}(x) = 0$  is possible but cumbersome. We prove the corresponding convergence using the comparison theorem. We consider only the case where  $a_n$  satisfies the Lipschitz condition. The general case follows by approximation.

It follows from the Itô–Tanaka formula that

$$\begin{aligned} d|X_n(t)| &= \text{sign}(X_n(t)) a_n(X_n(t)) dt + \text{sign}(X_n(t)) dW(t) + dl_n(t) \\ &= \text{sign}(X_n(t)) a_n(X_n(t)) dt + dW_n(t) + dl_n(t), \end{aligned}$$

where  $W_n$  is a new Wiener process, and  $l_n$  is the local time of  $X_n$  at zero.

Let  $-1/2 < c < \min(c_-, c_+, 0)$ . It follows from the arguments of [12] on comparison of reflecting SDEs that  $|X_n(t)| \geq Y_n(t)$ ,  $t \geq 0$ , where  $Y_n$  satisfies the following SDE with reflection at zero:

$$dY_n(t) = \bar{a}_n(Y_n(t)) dt + dW_n(t) + d\tilde{l}_n(t).$$

Here  $W_n(t) = \int_0^t \text{sign}(X_n(s)) dW(s)$  is a Wiener process,  $\tilde{t}_n$  is the local time of  $Y_n$  at 0,  $\bar{a}_n(x) = n\bar{a}(nx)$ ,  $\bar{a}(x) = -(|a(x)| + |a(-x)|) - \frac{c}{x} \mathbb{1}_{|x|>1} + r(x)$ , and  $r$  is any nonpositive function such that  $\bar{a}$  satisfies Lipschitz condition. We will also assume that  $\int_{\mathbb{R}} |r(x)| dx \leq \int_{\mathbb{R}} |b(x)| dx$ . The Lipschitz property is used only for application of comparison theorem.

To prove (19), it suffices to verify that

$$\lim_{\varepsilon \rightarrow 0} \sup_{x \in [0,1]} \sup_n \bar{u}_{n,\varepsilon}(x) := \lim_{\varepsilon \rightarrow 0} \sup_{x \in [0,1]} \sup_n \mathbb{E}_x \int_0^{\bar{\tau}_n} \mathbb{1}_{Y_n(s) \in [0,\varepsilon]} ds = 0,$$

where  $\bar{\tau}_n$  is the entry time of  $Y_n$  into  $[1, \infty)$ .

It is known [6] that

$$\bar{u}_{n,\varepsilon}(x) = 2 \int_x^1 \exp \left\{ -2 \int_1^y \bar{a}_n(z) dz \right\} \int_0^y \mathbb{1}_{[0,\varepsilon]}(z) \exp \left\{ 2 \int_1^y \bar{a}_n(z) dz \right\} dy$$

is a (generalized) solution of the equation

$$1/2 \bar{u}_{n,\varepsilon}''(x) + \bar{a}_n(x) \bar{u}_{n,\varepsilon}'(x) = -\mathbb{1}_{[0,\varepsilon]}, \quad x \in [0, 1],$$

with boundary conditions  $u_{n,\varepsilon}'(0) = 0$ ,  $u_{n,\varepsilon}(1) = 0$ . So

$$\begin{aligned} & \sup_{x \in [0,1]} \sup_n \bar{u}_{n,\varepsilon}(x) \\ &= \bar{u}_{n,\varepsilon}(0) \\ &= 2 \int_0^1 \exp \left\{ -2 \int_1^y \bar{a}_n(z) dz \right\} \int_0^y \mathbb{1}_{[0,\varepsilon]}(z) \exp \left\{ 2 \int_1^y \bar{a}_n(z) dz \right\} dy \\ &\leq K \int_0^1 \exp \left\{ \int_0^y \bar{y}^{-2c} dz \right\} \int_0^y \mathbb{1}_{[0,\varepsilon]}(z) y^{2c} dy, \end{aligned} \quad (21)$$

where  $K$  is a constant that depends only on  $\int_{\mathbb{R}} |b(x)| dx$  and  $c$  (and is independent of  $n$ ). By our choice,  $c \in (-1/2, 0)$ , so the right-hand side of (21) tends to 0 as  $\varepsilon \rightarrow 0+$  by the Lebesgue dominated convergence theorem.

The theorem is proved in cases A1–A5.

Consider case A6. Note that conditions (7)–(11) are satisfied for  $\xi^{(n)} = X_n$ ,  $n \geq 1$ , and  $\xi^{(0)} = X_\infty$ , where  $X_\infty$  is given in the theorem. In particular, this implies that the sequence of distributions of stochastic processes  $\{X_n\}$  in the space of continuous functions is weakly relatively compact. Choosing an arbitrary convergent subsequence, without loss of generality, we may assume that  $\{X_n\}$  itself converges weakly to a continuous process  $X$ . Let  $\delta > 0$ , and let  $\sigma^{n,\delta} = \inf\{t \geq 0: X_n(t) = \delta\}$ ,  $\sigma^\delta = \inf\{t \geq 0: X(t) = \delta\}$ . It follows from formulas for the scale function of the processes  $\{X_n\}$  that  $\lim_{n \rightarrow \infty} P(X_n(\sigma^{n,\delta}) = \delta) = p$ ,  $\lim_{n \rightarrow \infty} P(X_n(\sigma^{n,\delta}) = -\delta) = 1 - p$ , where  $p$  is given by (6). Formulas (9) and (11) imply that the limit process exits from the interval  $[-\delta, \delta]$  with probability 1.

Observe that, for almost all  $\delta > 0$ , with respect to the Lebesgue measure, the distribution of  $X_n(\sigma^{n,\delta} + \cdot)$  converges weakly as  $n \rightarrow \infty$  to the distribution of  $X(\delta + \cdot)$ . Indeed, by the Skorokhod theorem on a single probability space (see [16]), without

loss of generality, we may assume that the sequence  $\{X_n\}$  converges to  $X$  uniformly on compact sets with probability 1. For simplicity, we will assume that the convergence holds for all  $\omega$  and that also  $\sigma^{n,\delta}, \sigma^\delta < \infty$  for all  $\omega, n, \delta > 0$ . So we show convergence

$$X_n(\sigma^{n,\delta} + \cdot) \rightarrow X(\sigma^\delta + \cdot) \quad (22)$$

if we prove that

$$\sigma^{n,\delta} \rightarrow \sigma^\delta, \quad n \rightarrow \infty. \quad (23)$$

Convergence (23) may fail only if  $\sigma^\delta$  is a point of a local maximum of  $X$ . It follows from the definition that  $\sigma^\delta$  is a point of a strict local maximum of  $X$  from the left. The set of points of local maximums that are strict maximums from the left is at most countable. This yields that, for almost all  $\omega$  and almost all  $\delta > 0$  with respect to the Lebesgue measure, we have convergence (23) and hence (22).

On the other hand, the distribution of  $X_n(\sigma^{n,\delta} + \cdot)$  converges weakly as  $n \rightarrow \infty$  to the distribution of the process  $\mathbb{1}_{\Omega_-} B_{c_-}^-( -\delta, \cdot) + \mathbb{1}_{\Omega_+} B_{c_+}^+(\delta, \cdot)$ , where  $P(\Omega_-) = 1-p$ ,  $P(\Omega_+) = p$ , and the  $\sigma$ -algebra  $\{\emptyset, \Omega_-, \Omega_+, \Omega\}$  is independent of  $\sigma(B_{c_\pm}^\pm(\pm\delta, t), t \geq 0)$ .

Recall that assumptions of the theorem yield

$$P(\exists t \geq 0: B_{c_\pm}^\pm(\pm\delta, t) = 0) = 0.$$

It follows from (9) that

$$P\left(\int_0^\infty \mathbb{1}_{X(s)=0} ds = 0\right) = 1.$$

Thus, we have the almost sure convergence in  $C([0, \infty))$

$$X(\sigma^\delta + \cdot) \rightarrow X(\cdot), \quad \delta \rightarrow 0.$$

The processes  $\mathbb{1}_{\Omega_-} B_{c_-}^-( -\delta, \cdot) + \mathbb{1}_{\Omega_+} B_{c_+}^+(\delta, \cdot)$  converge in distribution to

$$\mathbb{1}_{\Omega_-} B_{c_-}^-(0, \cdot) + \mathbb{1}_{\Omega_+} B_{c_+}^+(0, \cdot),$$

where the  $\sigma$ -algebras  $\{\emptyset, \Omega_-, \Omega_+, \Omega\}$  and  $\sigma(B_{c_\pm}^\pm(0, t), t \geq 0)$  are independent.

This completes the proof of Theorem 1.

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